

## Cracking piles of brittle grains

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A model which accounts for cracking avalanches in piles of grains subject to external load is introduced and numerically simulated. The stress is stochastically transferred from higher layers to lower ones. Cracked areas exhibit various morphologies, depending on the degree of randomness in the packing and on the ductility of the grains. The external force necessary to continue the cracking process is constant in a wide range of values of the fraction of already cracked grains. If the grains are very brittle, the force fluctuations become periodic in early stages of cracking. The distribution of cracking avalanches obeys a power law with exponent  $\tau=2.4 \pm 0.1$ . [S1063-651X(99)04108-2]

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### I. INTRODUCTION

There are many phenomena concerning granular matter which have attracted the attention of physicists [1]. The source of the complexity of sand and similar systems stems from a highly nonlinear mechanical response on the mesoscopic scale (i.e., on the scale of single grains) which brings about complicated behavior on many scales, up to the macroscopic one, even though there is usually no scale-free behavior [2]. This feature brings the physics of granular matter close to other complex mechanics phenomena, like friction [3] and wear [4], where the interplay of mesoscopic and macroscopic phenomena is the central point of attention.

The dynamics of sand may be studied from two points of view. Slow driving by adding single grains gives rise to avalanches [5,6] and stratification phenomena [7]. Intense driving by periodic or persistent external forces was observed to cause, for example, surface pattern formation (dunes, etc.) or grain-size separation [1]. The dynamics of the mixture of sand and air may lead to beautiful phenomena like the ticking of hourglasses [8].

On the other hand, the most frequently asked question about static properties was the stress distribution within sand heaps, either free or embedded in various kinds of containers [2,9–13]. The most famous phenomenon is perhaps the minimum of stress directly below the top of a conic sandpile, measured by Smíd and Novosad [14] and later on explained theoretically by Bouchaud and co-workers [15–17]. The explanation is based on the fact that arches are created within the granular packing, which support most of the weight. A very important phenomenon connected with arching is the static avalanches due to large-scale reconstruction of arches, caused by very small external perturbation [18], and stick-slip motion of sand in a tube [19,20].

Both of the above phenomena are currently well described within the scalar arching model [19], which is a generalization of the scalar stress model developed for granular matter by Liu *et al.* [13,21,22].

A less studied phenomenon from the point of view of

granular materials is the procedure in which the grains are produced, i.e., the fragmentation process [23,24]. The obvious practical importance of this process was stressed, e.g., in [25]. In statistical approaches to fragmentation [23], the grains which are cracked are considered either independently of each other or random two-particle collisions of the grains are taken into account. Such models are appropriate to the situation in mills. Different mechanisms should be at work when the bulk of the heap of granular particles is cracked by compression, like in manufacturing pills in the pharmaceutical industry. Similar problems were already addressed when studying the localization of deformation in two-dimensional heaps of plastic cylinders [26] and compaction of granular matter in silos under pressure [27].

In the present work, we introduce a model which considers the cracking of grains within a pile of other grains, some of them already cracked, others not. So we will not investigate the size distribution of fragments, like in Ref. [23], but the spatial configuration of clusters of cracked grains and also the external force fluctuations occurring during the process of cracking.

The article is organized as follows. In the next section the model is introduced. The Sec. III is a gallery of simulation results and the last section, Sec. IV, draws conclusions from the results obtained.

### II. DESCRIPTION OF THE MODEL

Our model describes a two-dimensional pile of granular matter contained in a rectangular silo. A physical realization of this situation may be prepared by two parallel glass plates, the distance of which corresponds to the grain size. The lateral and bottom slots are closed, while the upper slot is open and a uniform external force is applied to the surface of the pile by a kind of piston. The grains are brittle (egg shells may serve as a popular example), which means that if the stress the grain supports exceeds a threshold value  $w_{\text{thr}}$ , the grain collapses. As a consequence of this, the stress pattern in the neighborhood of the collapsed grain changes, which may cause another grain collapse and finally leads to a kind of internal avalanche. During that process, the piston is kept immobile, so the total external force decreases, until the ava-

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lanche stops. How much the force decreases as a consequence of cracking one grain is described by a material-dependent factor  $\alpha < 1$ . We may expect that for more ductile grains the drop of the force will be smaller and the parameter  $\alpha$  will be closer to 1. For this reason we will call  $\alpha$  the ductility.

The stress within the pile is a tensor, but recent studies [17] showed that for many purposes only the diagonal element corresponding to the horizontal axis is important. This simplification leads to a scalar model of stress propagation in granular matter, which will be a basis of our model here.

We suppose the grains are placed regularly on a square lattice rotated by  $45^\circ$ , so that the columns and rows of grains correspond to the diagonals on the lattice. Each row is  $L$  grains wide; each column is  $H$  grains high. The grains are in contact with the nearest neighbors on the lattice. The randomness in the size, shape, and position of the grains is taken into account by a stochastic rule, which describes the propagation of stress.

Denote  $w_{ik}$  the stress on the grain in  $i$ th row (counted from above) and  $k$ th column. It transfers the fraction  $q_{ik}$  of the stress to its left bottom neighbor, the fraction  $1 - q_{ik}$  to its right bottom neighbor. We neglect the weight of the grains themselves, compared to the external force. So the rule of stress propagation is described by the equations

$$\begin{aligned} w_{i+1,k} &= q_{ik}w_{ik} + (1 - q_{ik-1})w_{ik-1} & \text{for odd } k, \\ w_{i+1,k} &= (1 - q_{ik})w_{ik} + q_{ik+1}w_{ik+1} & \text{for even } k. \end{aligned} \quad (1)$$

We impose cylindrical boundary conditions,  $w_{i0} = w_{iL}$ . The topmost row is subject to external forces  $w_{1k} = f_k$ . We will call the normalized sum  $F = \sum_k f_k / L$  the total external force.

The simulation proceeds as follows. The numbers  $q_{ik}$  are taken randomly from the uniform distribution on the interval  $((1 - \beta)/2, (1 + \beta)/2)$ . Initially all  $f_k$  are set equal and the local stresses are computed according to rules (1). At time step  $t$ , the force is uniformly increased until stress on one noncracked grain, say, at position  $(i, k)$ , reaches the threshold  $w_{\text{thr}} = 1$ . Then, the time is stopped and the cracking avalanche starts. The grain is cracked, which has two consequences.

First, the external force is lowered. We can introduce the response function  $G(i, k; k')$  such that the reduction of the external force on column  $k'$  is  $f_{k'} \rightarrow [1 - G(i, k; k')]f_{k'}$ . We suppose that the response is localized,  $G \sim \exp(-|k - k'|/\xi)$ , and the correlation length  $\xi$  is short,  $\xi \ll L$ . In this case we assume that the form  $G(i, k; k') = (1 - \alpha)\delta_{kk'}$  is a good approximation, which does not change the universality class of the model. This leads to a lowering of the force only on top of the column in which the cracked grain lies,  $f_k \rightarrow \alpha f_k$ .

Second, if grain in the same row to the left, i.e.,  $(i, k - 1)$ , is not cracked, the value of  $q$  corresponding to left top neighbor of  $(i, k)$  is set to 1. If  $(i, k - 1)$  is cracked,  $q$  is given a new random value from the uniform distribution on the interval  $((1 - \beta)/2, (1 + \beta)/2)$ . A similar rule applies on the right hand side: if  $(i, k + 1)$  is not cracked, the right top neighbor of  $(i, k)$  has a new  $q = 0$ ; if  $(i, k + 1)$  is cracked, the new  $q$  is a random number from the same distribution as above. These rules correspond to a very simple intuitive observation, that the cracked grain no longer bears the load, if it

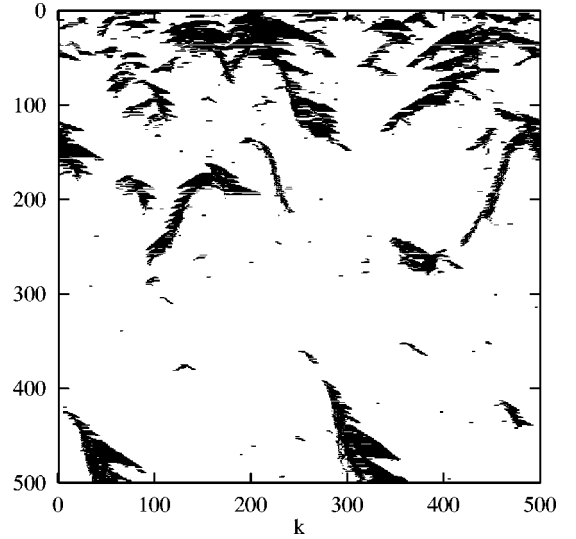


FIG. 1. Morphology of cracked areas for a sample with  $L = 500$  and  $H = 500$ , after 5000 time steps. Every cracked grain is represented by a black dot. The parameters are  $\alpha = 0.9$  and  $\beta = 0.25$ .

has neighbors, which can bear the load instead of it. However, if the neighbors are also already cracked grains, the stress propagation remains to be stochastic as it was before the cracking, but the realization of the randomness, i.e., the values of the numbers  $q$ , is changed.

After each change of  $q$ 's, the local stresses are recomputed, the grains which are not yet cracked and exceed the threshold are cracked, new  $q$ 's are established, and this procedure is repeated until no noncracked grains exceeding the threshold are found and the avalanche stops. Then we proceed to the next time step  $t + 1$ . The external force is increased up to the value when another grain is cracked again and a new cracking avalanche starts. We will call the avalanche size  $\Delta c$  the total number of grains cracked during the avalanche. This algorithm continues as long as there are any noncracked grains left.

Besides the size of the system, the model has two free parameters. The parameter  $\alpha$  measures the ductility of the grains and  $\beta$  the degree of randomness in the stress propagation. The limit  $\beta = 0$  corresponds to the fully deterministic case.

### III. SIMULATION RESULTS

When a grain is cracked, the load is mostly transferred to its neighbors, which have then increased their chances of being cracked. This leads to the creation of clusters of cracked grains, which grow and merge as the cracking proceeds. The typical morphology of the cracked clusters is shown in Fig. 1. We can observe the formation of ‘‘arches’’ with one dominant ‘‘leg’’ only. The shape of the ‘‘legs’’ resembles the letter S when they grow large. The dependence of the morphology on the ductility  $\alpha$  and randomness  $\beta$  is shown in Figs. 2, 3, and 4. For larger  $\beta$  the typical size of the cracked clusters is smaller, while for small  $\beta$  the sample contains only few big ‘‘arches,’’ which are also more symmetric than those for larger randomness. The ductility has a different influence on the morphology: in the case of more

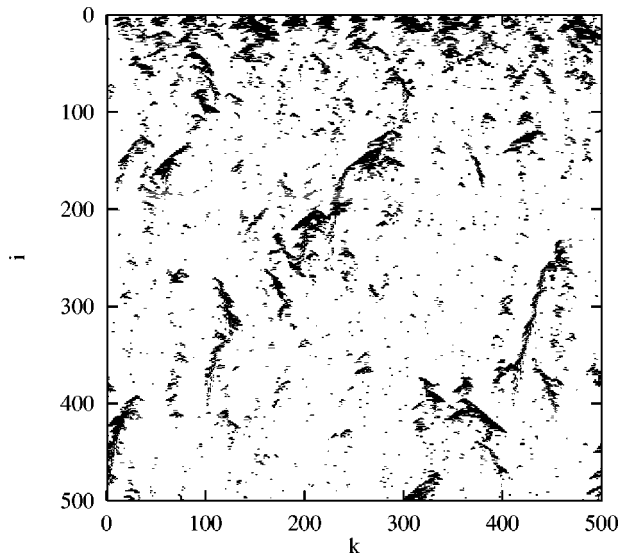


FIG. 2. Morphology of cracked areas for a sample with  $L=500$  and  $H=500$ , after 5000 time steps. The parameters are  $\alpha=0.9$  and  $\beta=0.5$ .

brittle grains, i.e., with smaller  $\alpha$ , the cracked areas are mostly concentrated in the top part of the sample, while more ductile grains lead to cracking equally probable in the whole bulk of the sample. (We performed simulations also for very ductile grains,  $\alpha$  close to 1, and the trend was observed to shift the cracked regions to the bottom of the sample, when the ductility is increased.)

When the cracking proceeds, the force necessary to continue fluctuates. Each cracking avalanche means a drop of the force, which then rises again. Figure 5 shows the time dependence of the external force  $F$  and the fraction of cracked grains  $\nu$  for a sample of  $200 \times 200$  grains. We can see that the force fluctuates around a nearly time-independent value  $F_{av} \approx 0.55$  during the large part of the process, at least from time  $t=1000$  to  $t=5000$ . This was even more clearly observed for larger samples (in our simulations

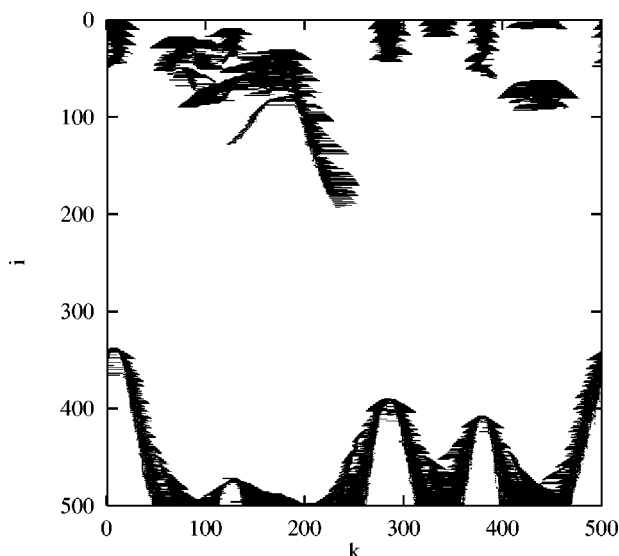


FIG. 3. Morphology of cracked areas for a sample with  $L=500$  and  $H=500$ , after 5000 time steps. The parameters are  $\alpha=0.9$  and  $\beta=0.1$ .

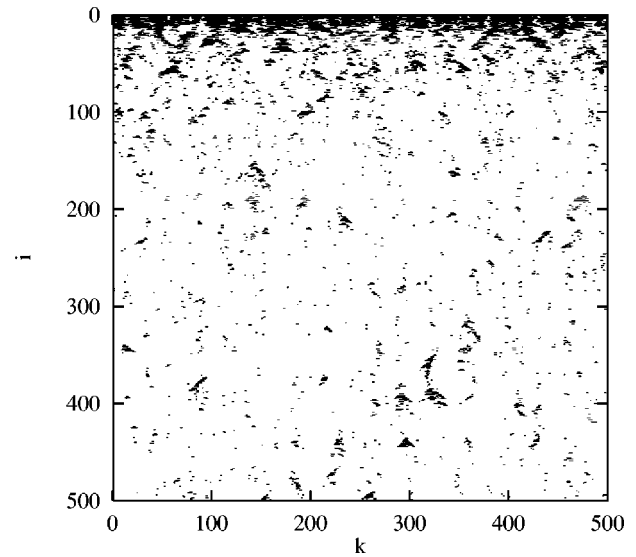


FIG. 4. Morphology of cracked areas for a sample with  $L=500$  and  $H=500$ , after 10000 time steps. The parameters are  $\alpha=0.5$  and  $\beta=0.5$ .

$500 \times 500$ ). So the picture of the overall behavior of the force can be as follows. After a transient period, where the force suddenly drops and slowly rises again, a stationary cracking regime develops, characterized by constant average force  $F_{av}$ . This regime holds if the fraction of cracked grains is small; according to our observations  $\nu < \nu_{max}$  is a sufficient condition, where the value of  $\nu_{max}$  depends slightly on  $\alpha$ . For  $\alpha=0.1$  we found  $\nu_{max} \approx 0.7$ , while for  $\alpha=0.9$  we observed  $\nu_{max} \approx 0.4$ .

The value of the stationary force  $F_{av}$  decreases with  $\beta$ . We found the values in the range from  $F_{av} \approx 0.3$  for  $\beta=1$  (maximum randomness) to  $F_{av} \approx 0.6$  for  $\beta=0.1$  (minimum randomness studied).

Around the average force, there are fluctuations which reflect the unique realization of the disorder in our sample.

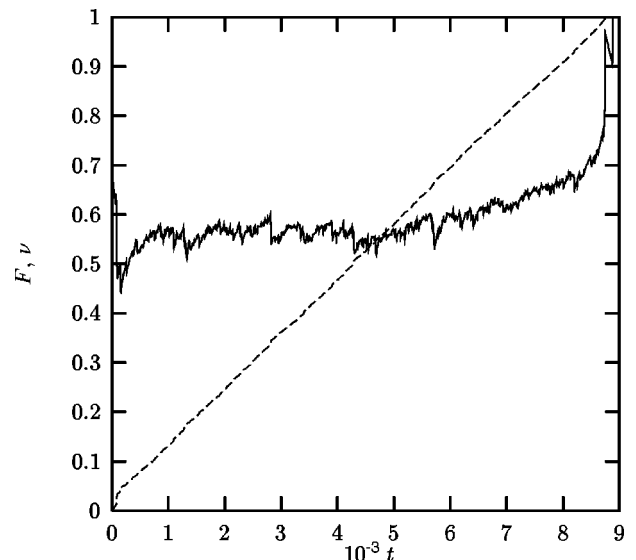


FIG. 5. Time evolution of external force  $F$  (solid line) and fraction of cracked grains  $\nu$  (dashed line) for the sample with  $L=200$ ,  $H=200$ ,  $\alpha=0.9$ , and  $\beta=0.25$ .

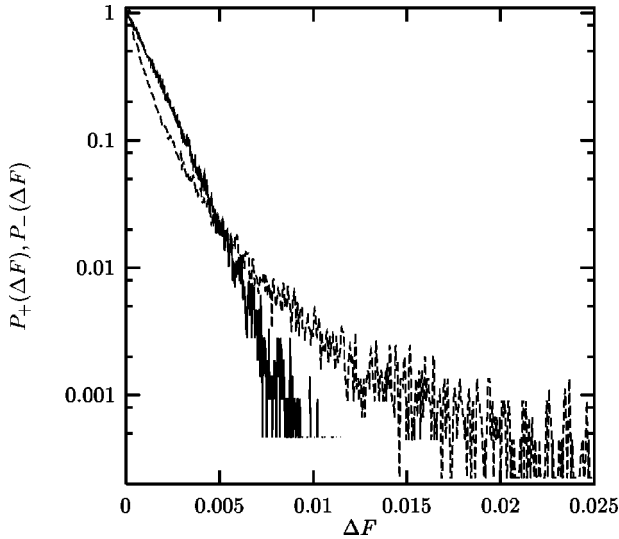


FIG. 6. Distribution of upward (solid line) and downward (dashed line) changes of the external force, for  $L=200$ ,  $H=200$ ,  $\alpha=0.9$ , and  $\beta=0.25$ . In order to avoid initial and final transient regimes, only the interval from time  $t=1000$  to  $t=8000$  was analyzed. The data are averaged over 20 independent runs.

We investigated statistical properties of the fluctuations. In Fig. 6 we show the distributions of upward changes  $P_+(\Delta F)$  and downward changes  $P_-(\Delta F)$  of the total external force from one step to the next one. The distribution of upward changes can be fitted well by an exponential, while the downward changes do not have any clear form of distribution: neither a Gaussian, exponential, stretched exponential, nor power-law fit was satisfactory. A distribution with a power-law tail seems to be a good candidate, but more data would be needed to settle this question.

For very small  $\alpha$  (we observed the phenomenon for  $\alpha=0.1$ , but for  $\alpha=0.3$  it was already absent) the fluctuations lose their purely random appearance and quasiregular force oscillations occur, which are especially pronounced in the early stages of the cracking process (i.e., for small  $\nu$ ). They can be clearly seen in Fig. 7. When the fraction of cracked grains increases, the oscillations gradually disappear. The oscillations perhaps correspond to the sudden drop of the force, observed for all  $\alpha$ , followed by a gradual increase of the force again. While for small  $\alpha$  many periods of the oscillation may be realized, for larger  $\alpha$  the oscillations are “over-damped” and only a single period occurs.

A cracking avalanche starts from the stable state, in which the stress on all noncracked grains is below the threshold. The avalanche is initiated by an increase of external force up to a value which causes one grain to crack. This cracking may result in the cracking of other grains, and so on, until a new stable state is reached and the avalanche stops. We denote  $\Delta c$  the avalanche size, which is the number of grains cracked during the avalanche. We are interested in a statistical distribution of avalanche sizes. We expect that the distribution may be different in the initial transient period and in the stationary regime, in which the average force  $F_{av}$  is constant. So we investigated the distributions  $P_l^>(\Delta c)$  defined as probabilities that the size of the avalanche, occurring in a time interval  $(t_{l-1}, t_l]$ , with  $t_0=0$ , is larger than  $\Delta c$ .

Figure 8 shows the results for a  $500 \times 500$  sample. The

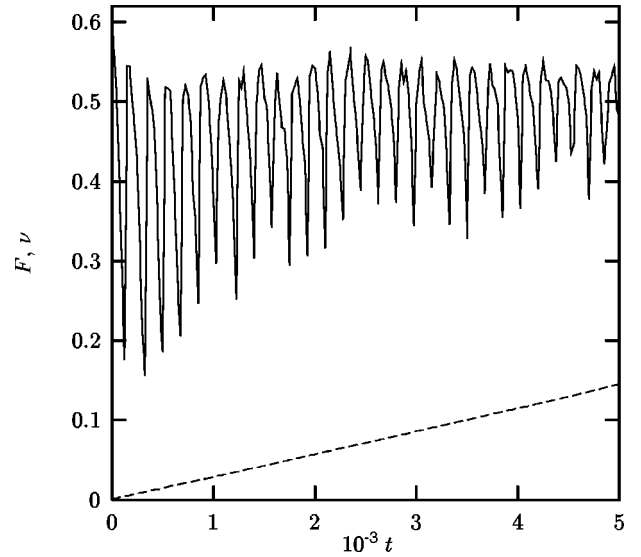


FIG. 7. Time evolution of external force  $F$  (solid line) and fraction of cracked grains  $\nu$  (dashed line) for the sample with  $L=200$ ,  $H=200$ ,  $\alpha=0.1$ , and  $\beta=0.25$ .

first two intervals, with final times  $t_1=100$  and  $t_2=300$ , describe the situation in the transition regime. We can see that most of the avalanches have a typical size of about  $\Delta c \approx 400$ . On the other hand, the next two intervals with end times  $t_3=1000$  and  $t_4=5000$  give distributions which can be fitted by a power law in the range of two decades. It can be also seen that the distribution is stable in time during the stationary cracking regime. We fitted the exponent of the power-law dependence  $P^>(\Delta c) \sim (\Delta c)^{1-\tau}$  with the result

$$\tau = 2.4 \pm 0.1. \quad (2)$$

We have found the same exponent (within error bars) for all values of the parameters studied. The only exception was

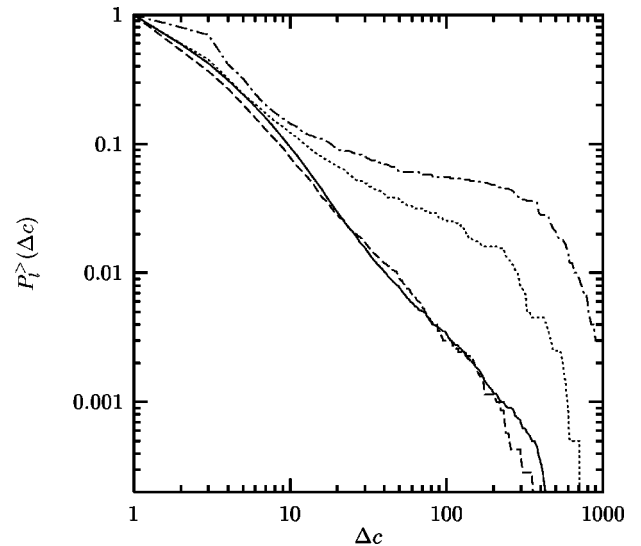


FIG. 8. Avalanche size distributions for  $L=500$ ,  $H=500$ ,  $\alpha=0.9$ , and  $\beta=0.25$ , in intervals determined by times  $t_1=100$ ,  $t_2=300$ ,  $t_3=1000$ , and  $t_4=5000$ . The lines denote the following distributions: dash-dotted line,  $P_1^>$ ; dotted line,  $P_2^>$ ; dashed line,  $P_3^>$ ; solid line,  $P_4^>$ .  $P_l^>$  corresponds to the interval  $(t_{l-1}, t_l]$ .



the case of  $\beta=1$ , where the distribution was close to exponential, instead of power law. The breakdown of the power law, when  $\beta$  approaches 1, remains to be studied.

#### IV. CONCLUSIONS

We have found that a two-dimensional pile of brittle grains packed in a rectangular container exhibits nontrivial behavior when an external force is applied from above and the grains are cracked. The cracked grains form clusters with different morphologies, depending on the ductility of the grains and on the degree of randomness in the packing. The degree of randomness seems only to determine the characteristic scale of the cracked clusters: lower randomness leads to larger clusters. This fact can be understood rather easily if we realize that a cluster occurs when the local stress exceeds the threshold necessary for a grain to be cracked. If the stress distribution is more uniform, fluctuations above the threshold are more distant one from the other.

A less expected feature is the influence of the ductility. Brittle grains have the tendency to crack in the top part of the container, while ductile grains are cracked mostly in the bottom part. This finding may play an important role in the separation of grains of different types.

During the cracking process the external force fluctuates around a general trend, which can be described as follows. If the grains are not too brittle ( $\alpha \geq 0.3$ ), the force drops suddenly and then rises slowly to a value which then remains constant for a great part of the whole cracking process. When the fraction of cracked grains approaches 1, the force increases again. So there exists a well-defined stationary cracking regime, preceded by a transient period and followed by a

final stage. For very brittle grains, the force oscillates rather regularly even in the stationary regime.

Cracking one grain may result in an avalanche of further crackings. The distribution of avalanche sizes depends on time. While in the transient period the distribution is not scale invariant, in the stationary regime the distribution of avalanche sizes obeys a power law. This is an indication that a sort of criticality is present in the cracking process. The value of the exponent  $\tau \approx 2.4$  is larger than the avalanche exponents found in most self-organized critical (SOC) models known to us, where typically  $\tau \leq 3/2$  [28–31]. On the other hand, the dynamics of our model resembles the Olami-Feder-Christensen (OFC) model of earthquakes [32], where the exponent varies in a wide range, comprising also the value found in our model. However, the mechanism leading to power-law scaling in the OFC model is not completely clear and the presence of SOC in that model is debated [33]. This may suggest that also in our model a new mechanism leading to criticality is at work, different from the usual SOC, represented by sandpile [28] or extremal dynamics [31] models.

This work does not compare the simulation results with experimental data, because we were not able to find any report of an experiment of this kind (loosely related are the experiments reported in [26]). It would be very welcome if a measurement in the direction suggested here was done in the future.

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